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NUMERICAL ANALYSIS RESEARCH  
UNPUBLISHED STATISTICAL TABLES

by  
D. Teichroew

A Technical Report

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Numerical Analysis Research  
Unpublished Statistical Tables<sup>1</sup>

by

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This note lists tables computed at what is now Numerical Analysis Research, University of California, Los Angeles and before July 1, 1954 was the Institute for Numerical Analysis, National Bureau of Standards.

Publication of tables is expensive and funds for the publication of statistical tables are extremely hard to find. It is therefore unlikely that many of the tables listed below will be published. Also, many of the tables are at present not ready for publication and some of the tables are too special to warrant publication. It seems worthwhile to report the existence of these tables and indicate their present state.

The following list contains tables connected with probability and statistics which have been computed at NAR. Some of these tables were computed as an end in themselves, others were by-products in computations or were computed because the machine codes were available from other problems. In general, the tables have not checked beyond the accuracy required for the purpose for which the table was intended. It seems worthwhile to list tables which are not completely verified because they may simplify the checking procedures if and when the functions are recomputed.

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<sup>1</sup>The preparation of this table was sponsored (in part) by the Office of Naval Research, USN and the School of Aviation Medicine, USAF.

The number of digits given in the tables may seem excessive to many statisticians. The number of digits used is a consequence of the fact that the tables were computed on the SWAC (Bureau of Standards Western Automatic Computer). This machine operates on a basic number length of 36 binary digits (10.8 decimal digits) and there is no point in using less than an integral multiple of the basic length and, in general, the tables retain most of the digits used in the computation.

Publication appears probable for the following three tables and they are therefore not listed elsewhere in this note.

1. "Tables of the bivariate normal distribution function and related functions." Collated by the National Bureau of Standards. The introduction is by G. Blanch.

The functions given are

$$L(h, k, r) = \frac{1}{2\pi \sqrt{1-r^2}} \int_h^\infty dx \int_k^\infty \exp \left[ -\frac{1}{2} \left( \frac{x^2 + y^2 - 2rxy}{1-r^2} \right) \right] dy$$

and

$$V(h, \lambda h) = \frac{1}{2\pi} \int_0^h dx \int_0^{\lambda x} \exp \left[ -\frac{1}{2}(x^2 + y^2) \right] dy .$$

2. "Table of salvo kill probabilities for square targets"

Applied Mathematics Series 44, National Bureau of Standards.

Introduction by A. D. Hestenes.

This table gives values of the function

$$P = \frac{1}{2\pi\sigma_{A_x}\sigma_{A_y}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(\xi, \eta) \exp \left[ -\frac{1}{2} \frac{(\xi - x_0)^2}{\sigma_{A_x}^2} - \frac{1}{2} \frac{(\eta - y_0)^2}{\sigma_{A_y}^2} \right] d\xi d\eta$$

where  $Q(\xi, \eta) = 1 - [1 - P_k P_R(\xi, \eta)]^N$ ;  $0 \leq P_k \leq 1$

and

$$P_R(\xi, \eta) = \frac{1}{2\pi\sigma_{R_x}\sigma_{R_y}} \int_{-a}^a \int_{-a}^a \exp \left[ -\frac{1}{2} \frac{(x - \xi)^2}{\sigma_{R_x}^2} - \frac{1}{2} \frac{(y - \eta)^2}{\sigma_{R_y}^2} \right] dx dy.$$

3. "Empirical power functions for non parametric two-sample tests for small samples" D. Teichroew. Submitted to a technical journal.

This paper gives the empirical frequencies of all possible rankings which are obtained when a sample of  $m$  from  $N(0, 1)$  and a sample of  $n$  from  $N(\delta, 1)$  are ranked in order of size, for  $(m, n) = (3, 2)$   $(3, 3)$   $(4, 2)$  and  $(4, 3)$  and various values of  $\delta$ .

All the tables listed below exist on punched cards. In addition, two tables mentioned in section II, namely, those containing  $y(p; \lambda)$  and  $Q_1(p)$ , have been multilithed and a limited number of copies are available for distribution.

The tables are listed under five categories:

- I. Tables associated with the normal distribution.
- II. Tables associated with the Gamma distribution.
- III. Tables associated with the t-distribution.
- IV. Tables for selecting samples from certain distributions.
- V. Miscellaneous tables.

## I. Tables Associated with the Normal Distribution

$$\text{Let } f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad F(x) = \int_{-\infty}^x f(t) dt,$$

$$H(x; b) = \int_{-\infty}^x [F(t)]^b dt,$$

$$\psi(a, b) = \int_{-\infty}^{\infty} [F(x)]^a \int_x^{\infty} [1 - F(y)]^b dy dx,$$

$$E(x_j; N) = \frac{N!}{(j-1)!(N-j)!} \int_{-\infty}^{\infty} x f(x) [F(x)]^{j-1} [1 - F(x)]^{N-j} dx,$$

$$E(x_j^2; N) = \frac{N!}{(j-1)!(N-j)!} \int_{-\infty}^{\infty} x^2 f(x) [F(x)]^{j-1} [1 - F(x)]^{N-j} dx,$$

$$E(x_i, x_j; N) = \frac{N!}{(i-1)!(j-i-1)!(N-j)!} G(i-1, N-j, j-i-1),$$

$$G(m, n, p) = \int_{-\infty}^{\infty} \int_{-\infty}^y xy f(x) f(y) [F(x)]^m [1 - F(y)]^n [F(y) - F(x)]^p dx dy,$$

and

$$K(\delta; \alpha, \beta) = (\beta + 1) \int_{-\infty}^{\infty} [F(x+\delta)]^{\alpha} [1 - F(x)]^{\beta} f(x) dx.$$



The range and size of the tables is as follows:

Function	Range	Decimal Places Tabulated	Accuracy
$f(x)$	$x = -12.00(.02)12.00$	32	$\geq 27$
$[F(x)]^k$	$x = -12.00(.02)12.00$ $k = 1(1)19$	32	$\geq 25$
$H(x; b)$	$x = -12.00(.02)12.00$ $b = 1(1)19$	32	$\geq 25$
$\psi(a, b)$	$a, b = 1(1)19$	32	$\geq 25$
$E(x_j, N)$	$j = 1(1)N$ $N = 1(1)22$	21	$\geq 19$
$E(x_j^2, N)$	$j = 1(1)N$ $N = 1(1)21$	21	$\geq 19$
$E(x_1, x_j, N)$	$i, j = 1(1)N$ $N = 1(1)20$	21	$\geq 19$
$K(\delta; \alpha, \beta)$	$\delta = -3.2(.1)0(.01)6.4$ $\alpha = 1(1)9$ $\beta = 0(1)4$	8	8

## II. Tables Associated with the Gamma Distribution

$$\text{Let } p = \frac{1}{\Gamma(\lambda)} \int_0^y (p; \lambda) e^{-t} t^{\lambda-1} dt .$$

Two SWAC routines have been developed.

1. The first computes  $p$  when  $y$  is given for  $\lambda$  an integer by summing the series

$$p(y, \lambda) = 1 - e^{-y} \left( 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots + \frac{y^{\lambda-1}}{(\lambda-1)!} \right)$$

The routine is also used to compute  $y$  for a given  $p$  by inverse interpolation.

2. The second routine computes  $y$  for a given  $p$  by the asymptotic series (using terms up to and including  $Q_{11}$ )

$$y(p; \lambda) = c^2 + Q_1(p) c + Q_2(p) + \frac{Q_3(p)}{c} + \dots$$

where  $\lambda = c^2$ .

These routines have been used to compute the following tables:

Function	Range	Decimal Places Tabulated	Accuracy
$p(y; \lambda)$	$y = 0(.5) \text{ varying; } \leq 45.0$ $\lambda = 2(1)15$	10	5-10
$y(p; \lambda)$	$p = .000(.001).999$ $\lambda = 2(1)15, 20(10)50, 100$	8	4-8
$2y(p; \lambda)$	$p = .000(.001).999$ $2\lambda = 3(1)5(2)29$	7	4-7
$Q_i(p)$	$p = .500(.001).999$ $i = 1(1)11$	8	8

### III. Tables Associated with the $t$ distribution

The tables are concerned with solving the equation

$$p = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n} \Gamma\left(\frac{n}{2}\right)} \int_{-\infty}^{t(p; n)} \frac{d\tau}{\left(1 + \frac{\tau^2}{n}\right)^{\frac{n+1}{2}}}$$

for  $t(p; n)$  when  $p$  and  $n$  are given. An approximation to  $t(p; n)$  is obtained by summing the first 8 terms of the asymptotic series

$$t(p; n) = x + \frac{H_1(p)}{n} + \frac{H_2(p)}{n^2} + \dots$$

The tables available are the following:

<u>Function</u>	<u>Range</u>	<u>Decimal Places Tabulated</u>	<u>Accuracy</u>
$H_i(p)$	$p = .500(.001).999$ $i = 1(1)8$	8	8
$t(p; n)$	$p = .500(.001).999$ $n = 2(1) 16, 20, 25, 50, 100, 200$	8	3-8

#### IV. Tables for Selecting Samples from Certain Distributions

One method of selecting samples from distributions consists of the following steps:

1. Compute,  $\theta(k)$ , the sum of  $k$  variates uniformly distributed on  $(0,1)$ .

2. Let  $y = \sum_{i=1}^m a_i \theta^i$ .

The  $a_i$  are chosen so that  $y$  has the required distribution. These coefficients have been computed for the case where  $k = 8$  and for the following distributions:

1. The normal distribution

2.  $\frac{t}{\sqrt{n}}$ ; where  $t$  has a normal distribution, for  $n = 50(1)200$

3. The Gamma distribution, for  $\lambda = 2(1) 15, 20, 25, 50, 100$ .

4.  $\frac{1}{\sqrt{y}}$  where  $y$  has a Gamma distribution for  
 $\lambda = 2(1) 15, 20, 25, 50, 100$ .

The variates  $\frac{t}{\sqrt{n}}$  and  $\frac{1}{\sqrt{y}}$  are used to generate random values of Wishart matrices.

## V. Miscellaneous Tables

## 1. Multinomial coefficients.

This table gives the function

$$\frac{(m + n + p + 2)!}{m! n! p!}$$

for all combinations of  $m$ ,  $n$ , and  $p$  such that  $m + n + p \leq 18$ .

The function occurs in the expression for the covariances of order statistics.

## 2. Coefficients for curve fitting by Chebyshev polynomials.

This table gives 10 decimal digit values of

$$\cos \frac{k\pi}{2n} (2\delta + 1)$$

for  $n = 2(2) 12, 16, 18, 20, 25, 30, 40$ ;  $k = 1(1) n - 1$ ,  $\delta = 0(1) n - 1$ .

This function appears in the curve fitting method described, for example, in "Tables of Chebyshev Polynomials  $S_n(x)$  and  $C_n(x)$ ," National Bureau of Standards, Applied Mathematics Series 9. Introduction by Cornelius Lanczos.

## 3. Tables for Probit Analysis with Poisson Error Models.

Three functions have been tabulated:

$$U(\alpha, \beta; h) = \sum_{d=1}^{\infty} \frac{e^{-h_d}}{d!} \int_{-\infty}^{\alpha + \beta \log_{10} d} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$$

$$Z(\alpha, \beta; h) = \sum_{d=1}^{\infty} \frac{e^{-h_d}}{d!} \frac{e^{-\frac{1}{2}(\alpha + \beta \log_{10} d)^2}}{\sqrt{2\pi}}$$

$$T(\alpha, \beta; h) = \sum_{d=1}^{\infty} \frac{e^{-h_d}}{d!} (\log_{10} d) \frac{e^{-\frac{1}{2}(\alpha + \beta \log_{10} d)^2}}{\sqrt{2\pi}}$$

for  $h = 1(1) 17$ ;  $\beta = 0(.5) 10$   $\alpha = -5(.5) 5$ . The accuracy varies from 3 to 8 decimal places.

December 8, 1954